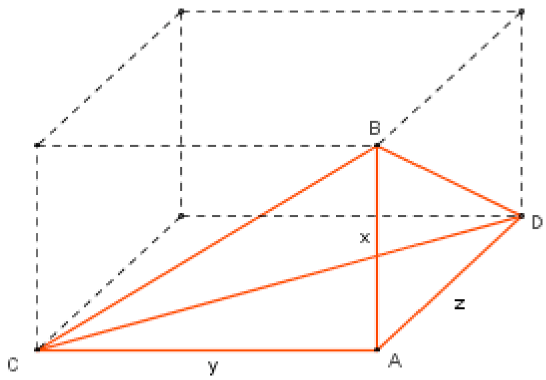
From Philippe Fondanaiche, Paris, France

Solution of the Challenge Problem 2009/2010 #04

In order to find a tetrahedron called "heronian" where the lengths of the edges, the area of the faces and the volume are integers, it is natural to consider in a first step special types of tetrahedra.

1) The first one is the right-angled vertex tetrahedron where in one vertex A all angles are right angles (see figure herefater)

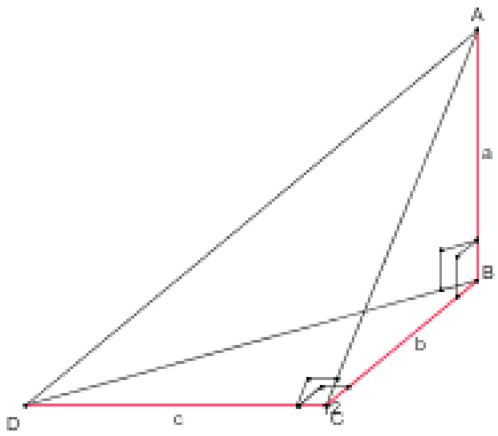


Let x = AB, y = AC and z = AD the lengths of the edges which end in A. It is easy to find integers x,y,z such that the lengths od the edges BC, BD and CD $\sqrt{x^2+y^2}$, $\sqrt{x^2+z^2}$ and $\sqrt{y^2+z^2}$ are also integer. Indeed x,y,z are the dimensions of an Euler brick (see http://mathworld.wolfram.com/EulerBrick.html). As ABC,ABD and ACD are pythagorean triangles, the areas of the faces which contain A are automatically integers. As at least one of the integers (x,y,z) is equal to 0 modulo 3, the volume of the tetrahedron equal to xyz/6 is also an integer.
The area of the face BCD (of semiperimeter p) opposite to A, calculated by the Heron formula, is equal to:

$\sqrt{p(p-\sqrt{x^2+y^2})(p-\sqrt{x^2+z^2})(p-\sqrt{y^2+z^2})}=\dfrac{\sqrt{x^2y^2+x^2z^2+y^2z^2}}{2}$ .This last term is an integer if and only if there is an integer box with the body diagonal equal to an integer.
Let (u,v,w) an integer box. Then with x = vw, y =uw and z = uv, the three edges $\sqrt{x^2+y^2}=w\sqrt{u^2+v^2}$, $\sqrt{x^2+z^2}=v\sqrt{u^2+w^2}$, $\sqrt{y^2+z^2}=u\sqrt{v^2+w^2}$ are integers and the area the face opposite to A is equal to

$uvw\dfrac{\sqrt{u^2+v^2+w^2}}{2}$ which is also an integer.Reciprocally, let u = yz, v = xz and w = xy the dimensions of a box defined with the edges AB,AC,AD of the tetrahedron.The lengths of the face diagonals of the box are $x\sqrt{y^2+z^2}$,$y\sqrt{x^2+z^2}$ and $z\sqrt{x^2+y^2}$ which are integers as well the body diagonal equal to $\sqrt{x^2y^2+x^2z^2+y^2z^2}$.
Conclusion: as the problem of an Euler brick which is an integer box is unsolved, there is no heronian tetrahedron with this first approach.

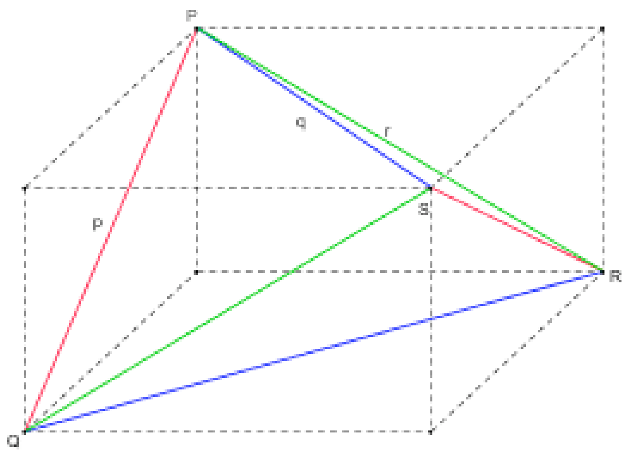
2) The second one is the right-angled-face tetrahedron in which all the faces are right-angles triangles (see figure hereafter).



Let AB = a, BC = b and CD = c.We have to find integer such that $\sqrt{a^2+b^2}$,$\sqrt{b^2+c^2}$ and $\sqrt{a^2+b^2+c^2}$ are integers.If it the case, the faces are pythagorean triangles and their areas are integers as well the volume of the tetrahedron equal to abc/6 (as in the first case).
A very simple computer program (see appendix 1 in VisualBasic) gives the smallest solution a= 153, b = 104, c = 672. Then AC = $\sqrt{a^2+b^2}$ = 185, BD = $\sqrt{b^2+c^2}$ = 680 and AD = $\sqrt{a^2+b^2+c^2}$ = 697. It is easy to check that all the areas of the faces and the volume of the tetrahedron are well integers.
Others solutions with (a,b,c) all <1000:

a	b	c
264	448	975
264	495	952
448	840	495
520	117	756

3) The third one is the tetrahedron inscibed in a box and whose opposite edges have the same length (see figure hereafter).



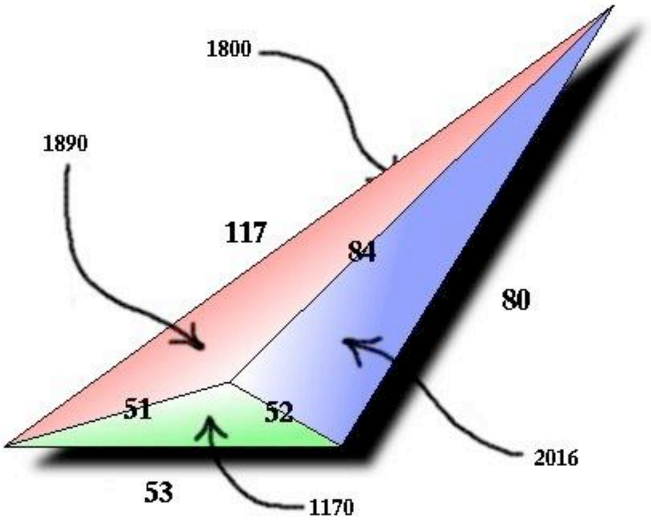
Let PQ = RS = p, QR = PS = q, PR = QS = r.

All the faces of the tetrahedron have the same area equal to A = $\frac{\sqrt{(p+q+r)(p-q+r)(p+q-r)(-p+q+r)}}{4}$ and the volume is equal to V = $\sqrt{\frac{(p^2+q^2-r^2)(p^2-q^2+r^2)(-p^2+q^2+r^2)}{72}}$.

Another simple computer programm (see appendix 2 in VisualBasic) gives the smallest solution p = 148, q = 195, r = 203, A= 13 650 and V = 611 520.

Other solution (multiples of the previous one omitted) with (pq,r) all <1000: (533, 875, 888).

In a second step, it is worth to look after tetahedra whose lengths are all different. There exist many sextuples (a,b,c,d,e,f) g such that the corresponding tetrahedron is heronian:
For example, the tetrahedron (51,53,52,117,84,80) illustrated hereafter by Ralph Buchholz (see http://sites.google.com/site/teufelpi/trailers/pp)



The areas of the faces are 1170, 1800, 1890, 2016 and the volume of the tetrahedron is equal 18144.
For more details,see:
http://mathworld.wolfram.com/HeronianTetrahedron.html
http://nova.newcastle.edu.au/vital/access/manager/Repository/uon:700/DS3

Appendix 1

```
Sub macrocase2
For a = 1 To 1000
For b = 1 To 1000
e1 = Sqr(a ^ 2 + b ^ 2)
If e1 = Int(e1) Then
For c = 1 To 1000
```

```
e2 = Sqr(b ^ 2 + c ^ 2)
If e2 = Int(e2) Then
e3 = Sqr(a ^ 2 + b ^ 2 + c ^ 2)
If e3 = Int(e3) Then
kk = kk + 1
Range("A" & kk).Value = a
Range("B" & kk).Value = b
Range("C" & kk).Value = c
End If
End If
Next c
End If
Next b
Next a
End Sub
```

Appendix 2

```
Sub macrocase3
For p = 1 To 500
For q = p To 500
For r = q To p + q - 1
a = Sqr((p + q + r) * (p + q - r) * (p - q + r) * (-p + q + r)) / 4
If a = Int(a) Then
v = (p ^ 2 - q ^ 2 + r ^ 2) * (p ^ 2 + q ^ 2 - r ^ 2) * (-p ^ 2 + q ^ 2 + r ^ 2) / 72
If v > 0 Then
v = Sqr(v)
If v = Int(v) Then
kk = kk + 1
Range("A" & kk).Value = p
Range("B" & kk).Value = q
Range("C" & kk).Value = r
Range("D" & kk).Value = a
Range("E" & kk).Value = v
End If
End If
End If
Next r
Next q
Next p
End Sub
```