Challenge problem:

Given x_k integer, $d_{n-1}\ldots d_1 d_0$, so $x_k=\sum_{i=0}^{n-1} d_i \times 10^i$ compute $x_{k+1}=f(x_k)$ with $f(x_k)=\sum_{i=0}^{n-1} d_i^p$, the power p fixed; a positive integer.

What happens with the series: x_o, x_1, x_2,.. for arbitrary positive x_0?

I)

Each series has a limiting periodic cycle, i.e. it ends in a cycle of the form:
$[x_k, x_{k+1}, \ldots, x_{k+q-1}]$ and $x_{k+q} = x_k$. With q the length of the period.

Proof:

$x_{k+1}=f(x_k) < \sum_{i=0}^{n-1} 9^p = n \cdot 9^p$ and $10^{n-1} <= x_k <= 10^n - 1$ (assuming n digits).
Since 10>9 the bounds on $x_k$will grow faster with n then the bound on x_{k+1}. So there exists a n_0, given p, for which $x_{k+1}<x_k$ if $n \geq n_0$. So starting with an x_0 with $n \geq n_0$ digits will result in a decreasing sequence of x_k. The remaining number of integers with n_0-1 digits or less are finite. Therefore ultimately $x_{k+q} = x_k$.

In the table below n_0 is calculated using the left bound on x_k.

p	9^p	n_0-1	x_k<=	x_{k+1}<=
2	81	3	999	243
3	729	4	9999	2916
4	6561	5	99999	32805
5	59049	6	999999	354294
6	531441	7	9999999	3720087
7	4782969	8	99999999	38263752

II)

We can use $(n_0-1)9^p$ as an upper bound for our search. For small p>1 use $(p+1)9^p$.

III)

Since the sum is indifferent to the order of the terms being summed, we know a permutation of the digits of x_k will lead to the same cycle.

So start with $(p+1)9^p$ bits all cleared, each representing an integer from 1 to $(p+1)9^p$. For each integer you want to use as x_o calculate the series x_k (if the bit for x_0 is not set) until you reach a x_l with $x_l=x_k$ for some k,l or until you find a x_k for which the bit is already set. In the latter case no new cycle is found. Clear all the bits of the x_k which are not set already (and it's permutations provided they are <= $(p+1)9^p$).

This claims procedure claims a lot off memory. Since I used Maple (and it has a lot of overhead) my limit was p=7. (Before swapping was necessary).

p=2: cycles

period 1: [1]
period 8: [4, 16, 37, 58, 89, **145**, 42, 20]

p=3

period 1: [1],[153],[370],[371],[407]
period 2: [919,**1459**],[136,244]
period 3: [55,250,133],[160,217,352]

p=4

period 1: [1],[1634],[8208],[9474]
period 2: [6514, 2178]
period 7: [**13139**, 6725, 4338, 4514, 1138, 4179, 9219]

p=5

period 1: [1],[4150],[4151], [54748], [92727], [93084], [**194979**]
period 2: [76438, 58618], [157596, 89883]
period 4: [10933, 59536, 73318, 50062]
period 6: [44155, 8299, 150898, 127711, 33649, 68335]
period 10: [92873, 108899, 183635, 44156, 12950, 62207, 24647, 26663, 23603, 8294]
[83633, 41273, 18107, 49577, 96812, 99626, 133682, 41063, 9044, 61097]
period 12: [24584, 37973, 93149, 119366, 74846, 59399, 180515, 39020, 59324, 63473, 26093, 67100]
period 22: [9045, 63198, 99837, 167916, 91410, 60075, 27708, 66414, 17601, 24585, 40074, 18855, 71787, 83190, 92061, 66858, 84213, 34068, 41811, 33795, 79467, 101463]
period 28: [70225, 19996, 184924, 93898, 183877, 99394, 178414, 51625, 14059, 63199, 126118, 40579, 80005, 35893, 95428, 95998, 213040, 1300, 244, 2080, 32800, 33043, 1753, 20176, 24616, 16609, 74602, 25639]

p=6

period 1: [1], [548834]
period 2: [313625, 63804]
period 3: [282595, 824963, 845130]
period 4: [93531, 548525, 313179, 650550]
period 10: [383890, 1057187, 513069, 594452, 570947, 786460, 477201, 239459, 1083396, 841700]
period 30: [301676, 211691, 578164, 446171, 172499, 1184692, 844403, 275161, 179996, **1758629**, 973580, 927588, 1189067, 957892, 1458364, 333347, 124661, 97474, 774931, 771565, 313205, 17148, 383891, 1057188, 657564, 246307, 169194, 1113636, 94773, 771564]

p=7

period 1 [1],[9926315],[1741725],[9800817],[4210818],[14459929]
period 2 [9057586, 8139850],[8807272, 5841646, 2767918],[6586433, 2755907]
period 6 [5345158, 2350099, 9646378, 8282107, 5018104, 2191663]
period 12 [6182897, 10080881, 6291458, 7254695, 6059210, 5141159, 4955606, 5515475, 1152428, 2191919, 14349038, 6917264]
period 14 [6896889, 16417266, 1679865, 8341662, 2675724, 2021787, 3744495, 5735976, 6868428, 6867840, 5594103, 4957791, 11307534, 922428]
period 21 [8543719, 7800361, 3202819, 6882565, 4910554, 4971988, 14600170, 1119865, 7238185, 5098288, 11152678, 3278887, 7940857, 8621716, 3480697, 8002171, 2920825, 6958630, 7520305, 982108, 8977402]
period 27 [7985787, 11526027, 1181862, 4474371, 1698426, 7456506, 1556049, 5235540, 253074, 920367, 5888763, 7475247, 2581650, 2533467, 1202490, 4799610, 10685802, 4552494, 4988499, 18575979, 13466427, 1418499, 11695860, 7518120, 2998950, 16524312, 376890]
period 30 [5905147, 5779147, 7348108, 5036419, 5159602, 5219284, 6974887, 10920679, 10669546, 5717287, 4646035, 672952, 5964829, 12037663, 1387918, 9803005, 6960433, 5363599, 10006498, 7176442, 1959919, **19210003**, 4785286, 5392420, 4879921, 12503146, 376762, 2209273, 5609083, 7240369]
period 56 [844431, 2148492, 6913146, 5361414, 393018, 6884496, 9569913, 14709156, 5980959, 16602309, 5345157, 1076490, 5902833, 6962748, 8280048, 6307968, 8265723, 3281199, 11665407, 1477926, 6726504, 1478052, 3015333, 86874, 5314167, 1200177, 1647216, 1399929, 19134192, 9584640, 7270950, 6508308, 4554552, 345396, 5161788, 5375910, 5764950, 6059082, 7238310, 2925198, 11741472, 1679985, 12844695, 7271079, 7253727, 2551197, 5762892, 8061981, 9257211, 5684895, 9429843, 11698173, 7985790, 13388301, 4200867, 3217143]
period 92 [5221343, 99140, 9582323, 6962876, 8543600, 2473784, 3779321, 6434558, 2568293, 7240625, 1198244, 6913019, 9848063, 9275780, 8605460, 2751533, 984296, 11959538, 11821529, 6958505, 7394432, 5643782, 3297455, 5781461, 3295142, 4879922, 12503273, 906299, 14628971, 8000114, 2113538, 2179781, 8527337, 3826865, 4834616, 2691980, 11943155, 4957793, 11309720, 5608829, 9335462, 5161916, 5420969, 9940511, 9660449, 10158578, 5174099, 10483991, 11681663, 2939150, 9646379, 10967924, 10685930, 7240370, 1665785, 3636818, 4758551, 3171455, 998366, 12225149, 4877864, 6154094, 5173799, 11293337, 5613203, 362564, 656696, 5980838, 11154737, 1743785, 3840935, 6979004, 10685801, 4552367, 1278428, 5034488, 4307384, 2957837, 8607647, 4320494, 4834436, 2430614, 315020, 80441, 2129921, 9566324, 5439665, 5517662, 1539794, 10486178, 5314169, 5159603]

Remarks:

Almost all these cycles can be discovered by limiting the search to x_0<=100! So for bigger p an educated guess to the cycles can be made by testing a small sample of starting values.

I used a bold typeface for the biggest number in any cycle for each p. This suggests that the upper bound can possibly be decreased. [Is there any way to predict this size?]